J. Fluid Mech. (1972), *vol.* 52, *part* 1, *pp.* 43–55
Printed in Great Britain

On a class of magneto-convective boundary-layer flows

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(Received 18 January 1971 and in revised form 23 July 1971)

A class of laminar rotating free-convective flows over a horizontal boundary in the presence of a strong magnetic field is investigated. The range of applicability of boundary-layer approximations is obtained, as well as a class of similarity solutions valid within a domain of multi-parameter space.

1. Introduction

Laminar, steady free-convective flows of an electrically conducting fluid in the presence of a magnetic field have been investigated extensively in recent years, e.g. Lykoudis (1962), Singh & Cowling (1963), Kuiken (1970). The multiplicity of physical parameters influencing such flows has made the derivation of boundary-layer solutions of more than cursory interest. It has, however, become increasingly clear that the domain of applicability of the boundary-layer simplifications needs careful scrutiny in each individual case. The flow considered here is that arising above a *horizontal* plate heated to a temperature above that of the surrounding fluid. The plate also rotates slowly at a steady angular velocity and in its own plane, while a strong magnetic field is aligned with the vector rotation. Schwiderski & Lugt (1967) have considered some such flows in the absence of buoyancy forces, while Rotem & Claassen (1969*a*, 1970) have considered the case without a magnetic field, both with and without rotation of the boundary. Several features were established for buoyant flows of this type: (i) The order of velocities found is smaller than for other free-convective flows. (ii) Boundary-layer flows are found to exist only under some well-defined special conditions. (iii) These flows have only a weak degree of stability against external disturbances and this is reflected in the numerical integration of the equations of motion governing the flow. Rotem & Claassen (1969*a*, *b*) also investigated some of these flows for the cases of vanishingly small and very large values of the Prandtl number σ and found that in both cases a double-layer structure results and necessitates the application of the method of matched asymptotic expansions. In those flows the process of heat transfer is governed by a balance of the forces of buoyancy, viscous drag and inertia; for small values of σ the forces of viscous drag will play only a second-order role, while as $\sigma \to \infty$ those arising from inertia will be of only secondary importance. In the present case forces arising from flow across a magnetic field will have to be included in the balance. Of

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particular interest will be the result that for fluids of large σ these latter forces are of negligible importance in the determination of the heat-transfer process.

2. The governing equations

The relations stating the conservation of mass, momentum, charge-flux density and energy are as follows:

$$\nabla.(\rho\mathbf{V}) = -\partial\rho/\partial t, \tag{1}$$

$$\frac{D\mathbf{V}}{Dt} = -\frac{\nabla p}{\rho}+\mathbf{g}+\frac{\mu}{\rho}\mathbf{J}\times\mathbf{H}+q\mathbf{E}+\nu\nabla^2\mathbf{V}+(\tfrac{1}{3}\nu)\nabla(\nabla.\mathbf{V}), \tag{2}$$

$$\mathbf{J} = (4\pi)^{-1}\nabla\times\mathbf{H} = \overline{\sigma}(\mathbf{E}+\mu\mathbf{V}\times\mathbf{H}), \tag{3}$$

$$\frac{DT}{Dt} = \frac{1}{\rho c_V}\nabla.(K\nabla T)-\frac{T}{\rho c_V}\frac{\partial p}{\partial T}\bigg|_\rho \nabla.\mathbf{V}+\frac{\nu}{c_V}\Phi_V. \tag{4}$$

In the above, ρ, μ, $\overline{\sigma}$, c_V and K are the scalar parameters density, susceptibility, kinematic viscosity, electrical conductivity, specific heat at constant volume and thermal conductivity respectively. $\mathbf{V}$, $\mathbf{g}$, $\mathbf{J}$, $\mathbf{H}$ and $\mathbf{E}$ are the velocity, gravity, current density, magnetic and electrical fields respectively, while q, t, T and Φ_V are the charge density, time, temperature and viscous dissipation function. To the equations above must be added an equation of state, relating the density to the other variables and parameters in the system. The following assumptions, which are entirely justifiable for the physical system in hand, will lead to a great simplification in the governing equations. (i) The flow is exactly steady and the fluid is Newtonian and incompressible, except in as far as the dependence of density upon the temperature is concerned. (ii) All the scalar parameters except ρ, as noted, are constants throughout the field. (iii) The distortion of the magnetic field is slight; displacement and magneto-convective currents may be disregarded. This assumption is liable to be rather restrictive in many cases of free-convective MHD flow but will be shown below to be less severe in the present case. (iv) Both the viscous dissipation Φ_V and Joule dissipation are negligible. (v) The flow field occupies a half-space with horizontal solid boundary; the bounding plane rotates around a normal axis with which the gravity and magnetic field vectors are aligned.

With assumptions (i) and (ii) above (which lead to the Boussinesq approximations) the equation of state becomes

$$\rho/\rho_\infty = 1-\epsilon(T-T_\infty)/T_\infty+O|\epsilon^2|, \tag{5}$$

where ϵ, the expansion coefficient at constant pressure, has a small numerical value. The subscript ∞ refers to values at a large distance from the boundary.

Clearly, under the assumptions made the flow is symmetrical around the axis of rotation, chosen as the z axis. Moreover, the configuration has no discernible characteristic length. This absence is a precondition for the existence of boundary-layer type approximations (cf. Kuiken & Rotem 1971). The flow variables will now be rendered dimensionless through the choice of a reference length L such that the largest value of any dimensionless co-ordinate be of order unity. The same procedure is followed for the magnetic field and for the temperature. The velocities arising in the case of buoyant flows over nearly horizontal surfaces are small in comparison with those over near vertical boundaries: their order is νL^{-1} (Rotem & Claassen 1969*b*). This will therefore be a suitable reference velocity to choose in the present case. With all these simplifications, and taking account of rotational symmetry, the governing equations become

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0, \tag{6}$$

$$u\frac{\partial u}{\partial r}-\frac{v^2}{r}+w\frac{\partial u}{\partial z}=-\frac{\partial \Pi}{\partial r}+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru)\right)+\frac{\partial^2 u}{\partial z^2}-M^2H^2u, \tag{7}$$

$$u\frac{\partial v}{\partial r}+\frac{uv}{r}+w\frac{\partial v}{\partial z}=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv)\right)+\frac{\partial^2 v}{\partial z^2}-M^2H^2v, \tag{8}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{\partial \Pi}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{\partial^2 w}{\partial z^2}\pm G\theta, \tag{9}$$

$$u\frac{\partial \theta}{\partial r}+w\frac{\partial \theta}{\partial z}=\frac{1}{\sigma}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right)+\frac{\partial^2 \theta}{\partial z^2}\right]. \tag{10}$$

In the above, all variables are dimensionless. u, v and w are the components of $\mathbf{V}$ in the radial, circumferential and normal directions respectively, θ is the dimensionless temperature normalized to a maximum value of unity and Π is the pressure,

$$\Pi=\frac{p-p_\infty}{\rho_\infty v^2}L^2+\frac{gL^3}{\nu^2}z.$$

G is the Grashof number $g\epsilon\Delta TL\nu^{-2}$, σ the Prandtl number ν/κ and M the Hartmann number $[\bar{\sigma}/(\rho\nu)]^{\frac{1}{2}}\mu LH$. The algebraical sign in (9) is positive for the case of the vector gravity acting along $-z$.

The equations (6)–(10) must be supplemented by suitable boundary conditions. These are

$$\left.\begin{aligned} r>0,\quad u=w=0,\quad v=Re\,G^{-\frac{2}{5}}r,\quad \theta=\theta(r,0),\quad \text{for}\quad z=0,\\ u=v=0,\quad \Pi=0,\quad \theta=0,\quad \text{for}\quad z\to\infty,\quad r\geqslant 0.\end{aligned}\right\} \tag{11}$$

Here Re is the Reynolds number $\Omega l^2/\nu$. With boundary conditions (11) the system of equations given is underdetermined but conditions on the vanishing of gradients of the velocity components far from the boundary furnish the missing boundary conditions. However, it will be shown that the conditions given in (11) are exactly sufficient for the solution of the boundary-layer equations.

In the case of free-convective flow along a nearly *vertical* surface in the presence of a magnetic field the buoyancy term would arise in (7) and (8). Then the qualitative ratio of magnetic forces to those of buoyancy is of order M^2/G. To render the terms of magnetic and buoyancy forces of comparable order in the equations of motion one would eliminate the parameter M^2G^{-1} through a suitable coordinate stretching; the natural reference velocity would readily be found to be $(M^2/G)(\nu/L)$, rather than ν/L as for the horizontal boundary. It is seen that even for the vertical boundary this velocity is small in comparison with that arising because of buoyancy in the *absence* of a magnetic field. Indeed, their ratio is of order $G^{\frac{1}{2}}M^{-2}$, a result also deducible from the work of Kuiken (1970). For the nearly vertical surface one is then immediately led to a boundary-layer transformation designed to retain only the largest of the viscous terms in the equations of motion, as follows:

$$\tilde{w} = (\sigma G M^{-2})^{\frac{1}{2}} w, \quad \text{etc.}$$

In the present case the terms due to buoyancy and those due to magnetic braking arise in different equations, coupling taking place *indirectly* through the pressure term. It is quite easily shown that the introduction of transformations along the lines outlined above leads to the loss of the buoyancy term and to the uncoupling of the equation of energy from the rest of the system. The correct boundary-layer transformation for the present case has been given by Rotem & Claassen (1970) and is as follows

$$\left.\begin{aligned} \tilde{z} = zG^{\frac{1}{5}}, \quad \tilde{u} = uG^{-\frac{2}{5}}, \quad \tilde{v} = G^{-\frac{2}{5}}, \\ \tilde{w} = wG^{\frac{1}{5}}, \quad \tilde{\Pi} = \Pi G^{-\frac{4}{5}}. \end{aligned}\right\} \tag{12}$$

These transformations lead to the correct retention of all terms in the equation of continuity, to the amplification of the small co-ordinate z to order unity in the transform plane and to the retention of the most important highest order terms in the equations of motion. For large values of the Grashof number they therefore lead to an 'inner' zeroth-order momentum boundary layer within which the forces of buoyancy, viscous friction and inertia are in balance. When the order of the Prandtl number is unity, it is within this very layer that the conductive and convective terms are in balance as well. Different considerations will apply when the Prandtl number takes on extreme values, as will be shown below. For other details of the theory the reader is referred to the various references.

The transformations (12) will now be inserted in the equations of motion and the assumption of $G \gg 1$ will be introduced. For convenience all superscripts have been dropped. The result is a set of boundary-layer equations in which the order of the highest order terms omitted has been indicated.

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{13}$$

$$u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} = \frac{\partial \Pi}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \left(\frac{M}{G^{\frac{1}{5}}}\right)^2 H^2 u + O|G^{-\frac{2}{5}}|, \tag{14}$$

$$u\frac{\partial v}{\partial r} + \frac{uv}{r} + w\frac{\partial w}{\partial z} = \frac{\partial^2 v}{\partial z^2} - \left(\frac{M}{G^{\frac{1}{5}}}\right)^2 H^2 v + O|G^{-\frac{2}{5}}|, \tag{15}$$

$$\frac{\partial \Pi}{\partial z} = \pm\theta + O|G^{-\frac{2}{5}}|, \tag{16}$$

$$u\frac{\partial \theta}{\partial r} + w\frac{\partial \theta}{\partial z} = \frac{1}{\sigma}\frac{\partial^2 \theta}{\partial z^2} + O|(\sigma^3 Ra^2)^{-\frac{1}{5}}|. \tag{17}$$

Here Ra is the Rayleigh number $g\epsilon\Delta T L^3/(\nu\kappa)$. The following qualitative conclusions may be drawn. (i) A measure of the ratio of magnetic to buoyant forces is the parameter $M^2G^{-\frac{2}{5}}$. (ii) A first-order correction to the temperature distribution will be of order $G^{-\frac{2}{5}}$, to the velocity distribution of order $G^{-\frac{1}{5}}$.

For the analysis outlined above to be applicable it is necessary for the Hartmann number to be of the same order as or smaller than $G^{+\frac{1}{5}}$; a singularity would result if the magnetic effect were overwhelmingly large.

We are now in a position to state the limitations on the orders of magnitude of the dimensionless parameters to which the boundary-layer equations are subject:

$$\left.\begin{array}{ll} G^{\frac{1}{5}} \gg 1, & MG^{-\frac{1}{5}} \leqslant 1, \\ \sigma \simeq 0|1|, & Re\,G^{-\frac{2}{5}} \leqslant 1. \end{array}\right\} \tag{18}$$

3. Similarity solutions

It has previously been shown (Rotem & Claassen 1970) that the existence of similarity solutions is limited to a temperature distribution such that $\theta(z = 0)$ increases with the square of the distance from the axis of rotation and such that there is no magnetic field. This will now be generalized to the case in which there is a magnetic field of the type $r^2\bar{H}(z)$. Introduce a Stokes streamfunction ψ,

$$u = \frac{1}{r}\frac{\partial \Psi}{\partial z}, \quad w = -\frac{1}{r}\frac{\partial \Psi}{\partial r}, \tag{19}$$

then the equation of continuity (13) is satisfied identically. Assume

$$\left.\begin{array}{c} \Psi = r^2F(\eta) \quad \text{with} \quad \eta = z, \\ \Pi = r^2P(\eta), \quad v = br\Gamma(\eta), \quad \theta = r^2S(\eta), \quad H = r^2\bar{H}(z), \end{array}\right\} \tag{20}$$

where b is the parameter $Re\,G^{-\frac{2}{5}}$. Thereupon the system (14)–(17) reduces to the following set of simultaneous quasi-linear ordinary differential equations:

$$F''' + 2F''F - F'^2 = 2P - b^2\Gamma^2 + (M/G^{\frac{1}{5}})^2F'\bar{H}^2, \tag{21}$$

$$P' = \pm S, \tag{22}$$

$$\Gamma'' + 2(F\Gamma' - F'\Gamma) - (M/G^{\frac{1}{5}})^2\Gamma\bar{H}^2 = 0, \tag{23}$$

$$S'' + 2\sigma(FS' - SF') = 0. \tag{24}$$

The boundary conditions subject to which these equations have to be solved are

$$\left.\begin{array}{llll} F = F' = 0, & S' = \Gamma = \bar{H} = 1, & \text{at} & \eta = 0, \\ F' = P = S = 0, & \bar{H}\ \text{specified}, & \text{as} & \eta \to \infty. \end{array}\right\} \tag{25}$$

The intensity of the magnetic field may be specified as a function of the z coordinate. For a constant magnetic-field function, $\bar{H}$ simply reduces to unity throughout.

4. Range of possible similarity solutions

The following lemma will now be proved.

LEMMA. For all possible similarity solutions the following holds:

$$-3\int_0^\infty\int_\zeta^\infty [F'(\omega)]^2\,d\omega \pm 2\int_\eta^\infty\int_\zeta^\infty\int_\omega^\infty S(\xi)\,d\xi\,d\omega\,d\zeta + b^2\int_\eta^\infty\int_\zeta^\infty [\Gamma(\omega)]^2\,d\omega\,d\zeta$$
$$-\gamma\int_0^\infty \{F(\infty) - F(\zeta)\}\,d\zeta \geqslant 0, \tag{26}$$

where $\gamma = M^2G^{-\frac{2}{5}}$.

Proof. For a constant magnetic field intensity of unity we obtain the following equation by differentiating (21) with respect to η.

$$F^{\mathrm{iv}} + 2F'''F = \pm 2S - 2b^2\Gamma\Gamma' + \gamma F''. \tag{27}$$

Integration from η to ∞ yields

$$-F''' - 2F'' + (F')^2 = \pm 2\int_\eta^\infty S(\zeta)\,d\zeta + b^2\Gamma^2 - \gamma F'. \tag{28}$$

Repeated integration from η to ∞ leads to

$$F'' - 2\int_\eta^\infty F''(\zeta)F(\zeta)\,d\zeta + \int_\eta^\infty [F'(\zeta)]^2\,d\zeta$$
$$= \pm 2\int_\eta^\infty\int_\eta^\infty S(\omega)\,d\omega\,d\zeta + b^2\int_\eta^\infty [\Gamma(\zeta)]^2\,d\zeta - \gamma F(\infty) + \gamma F(\eta),$$

or

$$F'' + 2FF' + 3\int_\eta^\infty [F'(\zeta)]^2\,d\zeta$$
$$= \pm 2\int_\eta^\infty\int_\zeta^\infty S(\omega)\,d\omega\,d\zeta + b^2\int_\eta^\infty [\Gamma(\zeta)^2\,d\zeta - \gamma F(\infty) + \gamma F(\eta), \tag{29}$$

whence

$$-F' + [F(\infty)]^2 - [F(\eta)]^2 + 3\int_\eta^\infty\int_\zeta^\infty [F'(\omega)]^2\,d\omega\,d\zeta$$
$$= \pm 2\int_\eta^\infty\int_\zeta^\infty\int_\omega^\infty S(\xi)\,d\xi\,d\omega\,d\zeta + b^2\int_\eta^\infty\int_\zeta^\infty [\Gamma(\omega)]^2 d\omega\,d\zeta$$
$$+\gamma\int_\eta^\infty [F(\eta) - F(\infty)]\,d\zeta. \tag{30}$$

The lower limit of integration will now be extended to $\eta = 0$. Noting that $F(0) = F'(0) = 0$ and that $F(\eta)$ must be real, one obtains (26), which completes the proof. The identity sign applies to the trivial case of no flow.

The proof given above is akin to a result for rather more general free-convective flows without either rotation or magnetic field discussed elsewhere (Rotem 1970). It is readily seen that whereas in the case of purely buoyant flows a similarity solution can exist only for a heated plate facing upwards (positive algebraical sign in front of the second term of (26)), in the present case a value of the parameter b larger than $\sqrt{2}$ will enable solutions in the case of a heated downward-facing rotating plate (or the alternative one of a cooled plate facing upwards). An investigation of the *existence* of these solutions does not form a part of the present investigation.

Three further conclusions may be drawn from inspection of (21), (22), (23) and (26): first, the obvious invariance of the equations of motion under reversal of the directions of rotation or of magnetic field; second, that the normal magnetic field acts always in a braking manner upon the layer of fluid nearest to the boundary and by implication leads to a more rapid thickening of the boundary layer; third, that a magnetic boundary layer will be present within which it is the magnetic and the inertia forces which balance, buoyancy playing a minor role.

5. Asymptotic cases of very large and very small σ

For the case of a very large value of the Prandtl number, the thermal boundary layer, i.e. that region of the flow over which the diffusive and advective terms in the energy equation are of the same order of magnitude, is rather thin in comparison with the momentum boundary layer. Indeed, the transfer of heat is almost entirely accomplished in the region of large viscous forces. The application of the limit $\sigma \to \infty$ to (17) would seem to lead to the loss of the diffusive terms in the energy equation and thus to singularity. Obviously the requirement is for a transformation which will eliminate σ as an independent parameter and which will retain these crucial terms in the energy equation.

Conversely, for vanishingly small values of σ the diffusive layer will extend beyond the momentum boundary layer; viscous forces will play only a small role over the extent of the thermal boundary layer. Mathematically it would seem that the value of the temperature at the boundary is operative throughout the momentum layer, whereas the energy equation is uncoupled from the momentum equations.

Prandtl number $\sigma \gg 1$

We shall introduce asymptotically 'stretched' variables which will yield the first term of an 'inner' solution (in σ) and transform the thickness of this layer to order unity. This is accomplished by the introduction of

$$\left.\begin{aligned} \tilde{z} = z\sigma^{\frac{1}{5}}, \quad \tilde{u} = u\sigma^{\frac{3}{5}}, \quad \tilde{w} = w\sigma^{\frac{4}{5}}, \\ \tilde{v} = v\sigma^{\frac{1}{10}}, \quad \tilde{\Pi} = \Pi\sigma^{\frac{1}{5}}. \end{aligned}\right\} \tag{31}$$

Equations analogous to (21)–(25) may now be derived:

$$\left.\begin{aligned} \tilde{F}''' - 2\tilde{P} + b^2 - \tilde{\gamma}\tilde{F}'\tilde{\tilde{H}}^2 &= O|\sigma^{-1}| = O|Ra^{-\frac{2}{5}}|, \\ \tilde{\Gamma}'' - \tilde{\gamma}\tilde{\Gamma}\tilde{\tilde{H}}^2 &= O|\sigma^{-1}| + O|Ra^{-\frac{2}{5}}|, \\ P' \pm S &= O|\sigma^{-\frac{2}{5}}| + O|Ra^{-\frac{4}{5}}|, \\ \tilde{S}'' + 2(\tilde{S}'\tilde{F} - \tilde{F}'\tilde{S}) &= O|Ra^{-\frac{2}{5}}|, \end{aligned}\right\} \tag{32}$$

where the constant $\tilde{\gamma}$ is $M^2Ra^{-\frac{2}{5}}$. Obviously, in the limit as $\sigma \to \infty$ there will be no influence of the magnetic field. On the other hand, for σ large but not infinitely large, the influence of the magnetic effects on the heat-transfer process will be of second order.

The 'inner' region, to which (32) apply, has to be supplemented by an 'outer' region, throughout which inertia and viscous forces balance with the magnetic force, buoyancy playing no role to first-order. The requisite stretching has been discussed elsewhere (Rotem & Claassen 1969*b*). It is shown there that the boundary conditions (25) will apply to the zeroth-order inner solution, with superscripted variables read for those of (25).

Prandtl number $\sigma \ll 1$

Proceeding as above, we notice that the heat-transfer process will here be governed by the 'outer' (Prandtl number) layer. The requisite transformations are

$$\left.\begin{aligned} \hat{z} = z\sigma^{\frac{2}{3}}, \quad \hat{r} = r\sigma^{\frac{2}{3}}, \quad \hat{u} = u\sigma^{\frac{1}{3}}, \\ \hat{v} = v\sigma^{\frac{1}{3}}, \quad \hat{w} = w\sigma^{\frac{1}{3}}, \quad \hat{\Pi} = \Pi\sigma^{\frac{2}{3}}, \end{aligned}\right\} \tag{33}$$

leading to the following system of equations:

$$\left.\begin{aligned} 2\hat{F}''\hat{F}-(\hat{F}')^2-2\hat{P}+b^2\hat{\Gamma}^2-\hat{\gamma}\hat{F}'\hat{H}^2 &= O|\sigma|+O|G^{-\frac{2}{5}}\sigma|, \\ \hat{\Gamma}'\hat{F}-\hat{F}'\hat{\Gamma}-\hat{\gamma}\hat{\Gamma}\hat{H}^2 &= O|\sigma|+O|G^{-\frac{2}{5}}\sigma|, \\ \hat{P}'\pm\hat{S} &= O|G^{-\frac{2}{5}}\sigma|-O|G^{-\frac{2}{5}}|, \\ \hat{S}''+2(\hat{S}'\hat{F}-\hat{F}'\hat{S}) &= O|G^{-\frac{2}{5}}\sigma^{\frac{2}{3}}|. \end{aligned}\right\} \quad (34)$$

Here the coefficient $\hat{\gamma}$ is $M^2G^{-\frac{2}{5}}\sigma^{-\frac{1}{3}}$. It is worth noting that the effect of the magnetic field is here greatly enhanced. As the highest order derivative in the momentum equations is now $\hat{F}''$ instead of a third-order term as for $\sigma \simeq O|1|$, it appears obvious that one of the boundary conditions previously used can no longer be fulfilled. Indeed, the condition of no-slip at the boundary, i.e. $F'(0) = 0$, is no longer assured. It is the function of a suitable 'inner' solution to ensure the disappearance of the velocity at the solid boundary. The stretching required has been given previously (Rotem & Claassen 1969*b*), and it has also been shown that the boundary conditions (25), with the one exception mentioned above and with superscripted variables substituted, are adequate to solve for the 'outer' layer. This layer here determines the heat transfer entirely.

6. Numerical integration

Equations (21)–(25) have been integrated for a variety of conditions: for buoyant flow with and without rotation, with and without magnetic field. Of particular interest is the approximate determination of the value of the constant b for which a breakdown of boundary-layer flow occurs;† this has also been obtained for one value of the Prandtl number.

σ	b	$F(\infty)$	$F''(0)$	$-F'''(0)$	$-P(0)$	$-P''(0)$ $= S'(0)$	$-\Gamma'(0)$
0·200	—	1·73420	1·89230	2·9532	1·50558	0·54599	
0·300	—	—	1·56174	2·5699	1·28497	0·60807	
0·500	—	1·01923	1·28175	—	1·11081	0·69312	
0·720	—	0·83537	1·09657	1·84908	0·99343	0·75702	
0·980	—	0·70951	0·96170	1·81552	0·90776	0·81381	No
1·000	—	0·70521	0·95343	1·80484	0·90242	0·81767	rotation
2·000	—	0·51191	0·71307	—	0·74709	0·95700	
5·000	—	0·35821	0·49025	—	0·59513	1·17045	
10·000	—	0·28160	0·37107	—	0·50686	1·35820	
20·000	—	0·22424	0·28136	—	0·43450	1·57220	
0·720	1	0·85071	1·35506	—	0·95833	0·79460	0·92318
0·720	5	1·08337	6·02628	—	0·67513	1·20707	1·41908
0·720	9	1·36209	13·96361	—	0·52308	1·58032	1·86201
1·000	1	0·72707	1·21751	—	0·85717	0·87068	0·87068
1·000	5	1·03114	5·94604	—	0·55389	1·40464	1·40464
1·000	9	1·34088	13·91391	—	0·42107	1·85665	1·85665
10·000	1	0·45799	0·68786	—	0·38058	1·75608	0·65541
10·000	5	0·98961	5·74464	—	0·18263	3·63030	1·37914
10·000	9	1·32694	13·79858	—	0·13629	4·86375	1·84835

TABLE 1. No magnetic field, heated surface upwards

† This coincides with the vanishing of $F''(0)$ for the case of the heated downward facing plate (or equivalently cooled upward-facing).

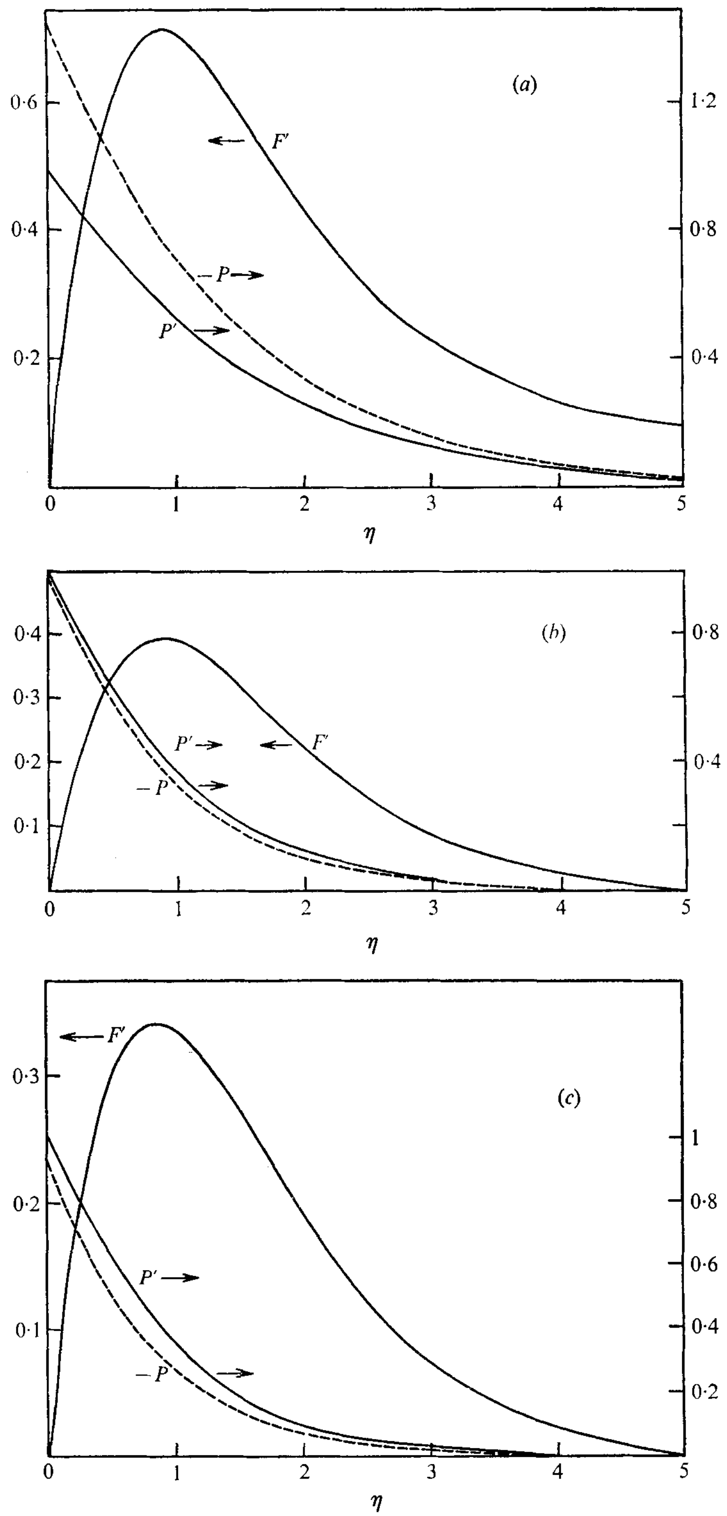
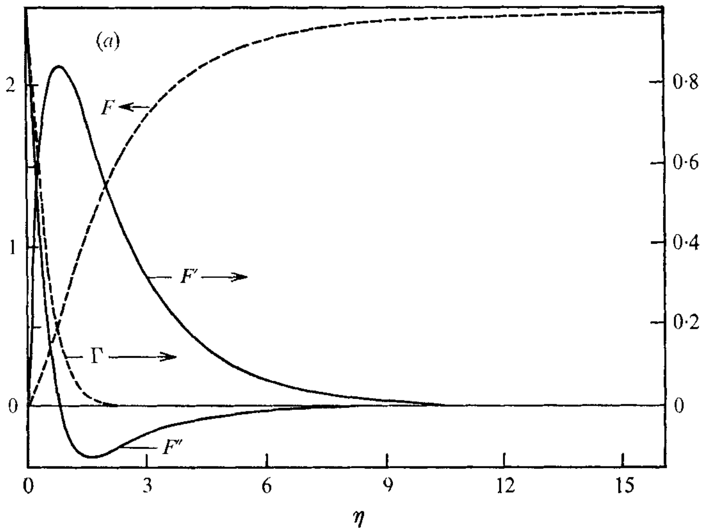


FIGURE 1. Velocity, pressure and temperature functions, $b = 0$, $\gamma = 0$, $\sigma = 0{\cdot}20$, $0{\cdot}7$, $0{\cdot}98$ for (*a*), (*b*), (*c*) respectively.



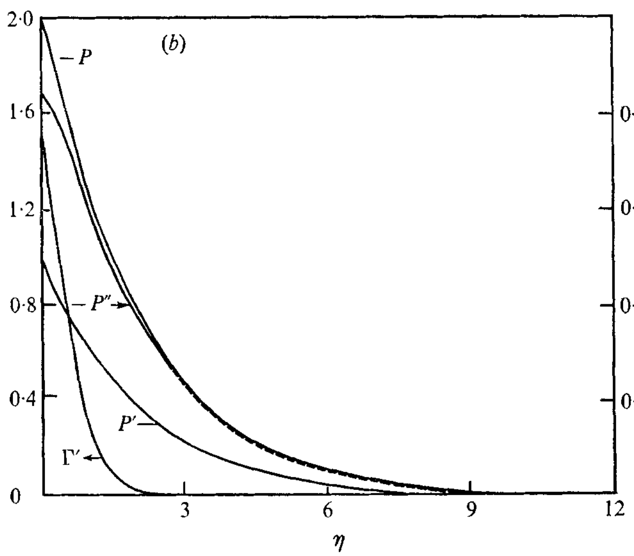


FIGURE 2. Computed values of functions for $\sigma = 0{\cdot}10$, $b = 1{\cdot}0$, $\gamma = 1{\cdot}0$, (*a*) F, F', Γ, (*b*) P, P', Γ'.

As already mentioned, the system of simultaneous equations proves very difficult to integrate; close approximations to the final values of the missing initial conditions are required in order to obtain a stable iterative scheme. The range of integration in the independent variable η necessary in order to simulate conditions at infinity correctly depends on the Prandtl number. This is evident from the stretching transformations (31) and (33).

Both the method of integration described by Rotem & Claassen (1969*a*) and an extensively modified scheme based upon the work of Swigert & Nachtsheim (1965) were used in conjunction with a fourth-order predictor corrector method. Some results are given in figures 1 and 2. Data which have to be known accurately

σ	b	$b^{-\frac{1}{2}}F(\infty)$	$b^{-\frac{3}{2}}F''(0)$	$-b^{-\frac{1}{2}}\Gamma'(0)$	$-b^{\frac{1}{2}}P(0)$	$-b^{-\frac{1}{2}}P''(0)$
10·0	4·9	—	0·50626	0·61491	0·40985	1·61709
10·0	4·0	—	0·50373	0·61430	0·41037	1·61499
10·0	3·0	—	0·49702	0·61270	0·41159	1·60940
10·0	2·0	—	0·47349	0·60699	0·41614	1·58949
10·0	1·5	—	0·43241	0·59674	0·42460	1·55355
10·0	1·0	—	0·25679	0·54813	0·47022	1·37898
10·0	0·8839	0·38555	0·03656	0·46834	0·57272	1·07544

TABLE 2. No magnetic field; 'unstable' configuration, heated surface downwards. $F''(0)$ vanishes at about $b = 0·8834$; thus there is no valid boundary-layer flow for lower values

σ	b	$(MG^{-\frac{1}{5}})^2$	$\bar{H}$	$F(\infty)$	$F''(0)$	$-\Gamma'(0)$	$-P(0)$	$-P''(0) = -S'(0)$
0·10	1	0·010	1	2·63078	2·72822	1·28538	1·92432	0·45077
0·72	1	0·010	1	0·84900	1·35358	0·92711	0·95958	0·79387
1·00	1	0·010	1	0·72515	1·21611	0·87491	0·85842	0·86981
10·00	1	0·010	1	0·45391	0·68560	0·66017	0·38154	1·75259
0·10	5	0·010	1	2·70185	6·98922	1·59727	1·83188	0·50989
0·72	5	0·010	1	1·08145	6·02005	1·42120	0·67599	1·20603
1·00	5	0·010	1	1·02899	5·93963	1·40674	0·55459	1·40344
10·00	5	0·010	1	0·98719	5·73777	1·38120	0·18276	3·62813
0·10	1	0·500	1	2·52925	2·60484	1·40657	2·00034	0·43572
0·72	1	0·500	1	0·77713	1·29162	1·11207	1·01792	0·76170
1·00	1	0·500	1	0·64799	1·15812	1·07227	0·91553	0·83246
10·00	1	0·500	1	0·27656	0·60398	0·89397	0·42727	1·60926
0·10	5	0·500	1	2·60191	6·71914	1·68815	1·90295	0·49298
0·72	5	0·500	1	0·99341	5·72924	1·52531	0·71987	1·15599
1·00	5	0·500	1	0·93004	5·64030	1·51051	0·59063	1·34574
10·00	5	0·500	1	0·87234	5·41467	1·48258	0·18930	3·52249
0·10	1	1·000	1	2·43369	2·50059	1·52930	2·07671	0·42180
0·72	1	1·000	1	0·72185	1·24374	1·28554	1·07186	0·73479
1·00	1	1·000	1	0·59380	1·11461	1·25450	0·96677	0·80228
10·00	1	1·000	1	0·19242	0·56061	1·11366	0·46637	1·51466
0·10	5	1·000	1	2·50459	6·46665	1·78184	1·97564	0·47677
0·72	5	1·000	1	0·91516	5·46124	1·63205	0·76723	1·10731
1·00	5	1·000	1	0·84214	5·36442	1·61726	0·63069	1·28893
10·00	5	1·000	1	0·76553	5·11193	1·58709	0·19639	3·41699

TABLE 3. Flow with rotation and magnetic field heated face upwards

are given in tables 1, 2 and 3. The intensity of the heat transfer, characterized by the Nusselt number Nu, is directly obtainable from the numerical results given in the tables.

$$Nu = -(\partial\theta/\partial\eta)_{\eta=0}G^{\frac{1}{5}} = -r^2S'(0)G^{\frac{1}{5}}. \tag{35}$$

7. Conclusions

In the course of the present analysis it has been found possible to delimit a range of applicability of the boundary-layer approximations within which the flow in combined free- and forced-convection in the presence of a magnetic field could be obtained. It was found that the boundary-layer range was severely limited by the inequalities (18) (for Prandtl numbers of order unity). For large

Prandtl numbers the magnetic field had little influence on the transport process, while its effect was greatly enhanced for fluids of small values of σ. Thus, while magnetic effects on the flow of aqueous solutions of electrolytes may be safely discounted, they may be crucial in the flow of liquid metals. In the latter case the range of free-convective boundary-layer flow will extend to values of G considerably lower than for other fluids.

The flow obtained was found to depend upon the parameters $G^{\frac{1}{5}}$, $MG^{-\frac{1}{5}}$, σ, $Re\,G^{-\frac{2}{5}}$ and b.† The possible solutions thus trace out a hypersurface in a multidimensional parameter space, being, however, limited to particular domains through both the inequalities (18) (their analogues for small and large values of σ respectively) and the integral theorem (26). Obviously, solutions of the full equations of motion, if they could be obtained, would not be subject to such a limitation. It is interesting to note that the flows obtained are invariant under reversal of both the sense of rotation and the direction of the magnetic field, but not a reversal of the vector gravity.

It seems possible that theorems of this kind should be obtainable for a variety of other magneto-convective flows. This appears to be a subject worthy of further investigation.

All solutions given in the present work are partial in the sense that zeroth-order approximations to either the inner or the outer regions alone were investigated in detail. It was shown that expansion solutions of the boundary-layer type will show multiple singularities, each of which has (in principle) to be solved by its own inner and outer expansion scheme. The main features of the heat-transport process are however given here; further refinements will add to our understanding of the flow process without substantially modifying the results given.

The type of flow investigated is entirely within the range of practical experiment. A sample calculation for liquid sodium yields the following numerical results. The density at 470 °K is approximately $0{\cdot}90\,\mathrm{g\,cm^{-3}}$, the expansion coefficient at constant pressure $2{\cdot}5\times10^{-4}\,{}^{\circ}\mathrm{K}^{-1}$, the kinematic viscosity $0{\cdot}5\times10^{-2}$ Stokes, the electrical conductivity $0{\cdot}73\times10^{-10}\,\mathrm{sec\,cm^{-2}}$. The value of the Prandtl number is 7×10^{-3} so, for a maximum temperature difference of the surface from its surroundings of 6·3 °K and for a disk of 20 cm diameter, the Grashof number is 5×10^{8}. The Hartmann number should therefore be at most about 55. With $MG^{-\frac{1}{5}}=\frac{1}{3}$, this yields a typical induction of 7430 gauss. Also, the order of the rotational Reynolds number is 3030, whence a typical rotational speed of the disk is found to be about $\frac{1}{2}$ rev/min.

Thanks are due to the Canada Department of Transport, Meteorological Branch and to the Canada National Research Council for financial support of this research work.

† To this has to be added the functional dependence of $\bar{H}$ upon z, i.e. if, for example, $\bar{H}(z) = \exp(-nz)$ then n will be an extra free parameter.

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